

A Crash Course in Logic, for Students of Mathematics

In a geometry course you need to understand and construct proofs, and for this you need to be able to analyze the logical structure of the statements involved. This analysis can be done at various levels of formality, with varying level of precision. The purpose of this brief outline is to give you the tools needed to analyze statements with more precision than ordinary English statements can easily support. Learning a few specialized terms, symbols, and ideas will repay your effort generously.

Mathematics is concerned with both propositional logic and predicate logic. Basically, propositional logic uses as building blocks complete statements which are either true or false. Such statements are called propositions.

Propositions:

- A: Socrates is a man
- B: All men are mortal
- C: Socrates is a cat
- D: All men are cats

We can build up compound statements from these simple statements, or *atoms*. The truth value of the atoms determines the truth value of the compound statements, according to these rules:

Symbols	Pronunciation	Truth value	Name
$P \& Q$	“P and Q”	is true only if P is true and Q is also true	Conjunction
$P \text{ or } Q$	“P or Q”	is true if either P, or Q, or both, are true	Disjunction
$P \Rightarrow Q$	“P implies Q” “if P, then Q”	is true if Q is true, or P is false, or both	Implication, or Conditional

These rules are summarized in this table, called a *truth table*:

P	Q	$P \& Q$	$P \text{ or } Q$	$P \Rightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

The tricky thing is the truth value of the implication $P \Rightarrow Q$. Historically there was a lot of debate about whether this statement should be considered true or false if the hypothesis (P) was false. Bertrand Russell won the debate, and in mathematics the convention is that we call the implication true in that case. (In other fields, this convention does not necessarily hold sway.)

Hence a statement like “If that’s a Mercedes, then I’m Elvis Presley,” is automatically true if “that” is not a Mercedes, regardless of whether the speaker is or is not Elvis Presley.

Any proposition can be negated. $\sim P$ (“not P”) is true if and only if P is false. When you negate compound statements, you get a statement which is true if and only if the original compound statement is false.

Exercise: Construct a version of the “truth table” above with additional columns for $\sim P$, $\sim Q$, $(\sim P \ \& \ \sim Q)$, $(\sim P \ \text{or} \ \sim Q)$, $(\sim P \ \text{or} \ Q)$, and $(P \ \& \ \sim Q)$. Verify from this table that:

- $\sim(P \ \& \ Q)$ always has the same truth value as $(\sim P \ \text{or} \ \sim Q)$
- $\sim(P \ \text{or} \ Q)$ always has the same truth value as $(\sim P \ \& \ \sim Q)$
- $(P \Rightarrow Q)$ always has the same truth value as $(\sim P \ \text{or} \ Q)$
- $\sim(P \Rightarrow Q)$ always has the same truth value as $(P \ \& \ \sim Q)$

So, the negation of a conditional statement $P \Rightarrow Q$ is the conjunction $(P \ \& \ \sim Q)$.

Two statements are called *equivalent* if they have the same truth value in all circumstances.

Every *implication*, or conditional statement, has three close relatives – conditional statements built from the same basic components.

Name	Symbolic Form	Example	Truth Value
Original Statement	$A \Rightarrow B$	If it rains, I’ll get wet.	Let’s assume true (I have no umbrella).
Contrapositive	$\sim B \Rightarrow \sim A$	If I don’t get wet, it isn’t raining.	Always the same as original statement.
Converse	$B \Rightarrow A$	If I get wet, it is raining.	I could be swimming. Truth value unrelated to truth value of original statement.
Inverse	$\sim A \Rightarrow \sim B$	If it doesn’t rain, I won’t get wet.	Same truth value as the converse.

Exercise: Translate each of the following statements into symbolic form, write the converse, inverse, and contrapositive in symbolic form, and translate those statements to English. Decide the truth value of each of the four statements (making reasonable assumptions, as necessary). Verify that the contrapositive always has the same truth value as the original, and that the inverse always has the same truth value as the converse.

- If I don’t pass this class, I won’t graduate.
- If Fred is in Paris right now, he isn’t in New York.
- If you play in the NFL, you are an athlete.

In mathematics, we often have complicated structures, but some parts are taken for granted. For instance, Euclid's fifth postulate says "If two lines are crossed by a transversal, and the interior angles on one side of the transversal sum to less than 180 degrees, then the lines meet on that side of the transversal." What is the inverse of this statement? Several plausible beginnings lead to nothing of mathematical interest.

- If two lines are not crossed by a transversal..... (then we can't even talk about the angles)
- If there aren't two lines.... (this is even less productive!)

Probably the most useful version of the inverse comes from taking some of the setup for granted, and excluding it from the conditional statement. So suppose that we have a situation where two lines l and m are crossed by a transversal t . Then Euclid's Fifth becomes

- If the interior angles on one side of t sum to less than 180 degrees, then l and m meet on that side of t . (This is Euclid's Vth Postulate, assumed true in Euclidean geometry.)

The inverse of this conditional statement (still assuming that two lines l and m are crossed by a transversal t) is then

- If the interior angles on one side of t sum to 180 degrees or more, then l and m do not meet on that side of t . (Do **not** assume that this statement is true! See following paragraph.)

In Euclidean geometry we take the original statement (Euclid V) as true. Whether the inverse is true in Euclidean geometry is, at this point, an open question. We may be able to prove the inverse, in which case it is a theorem of Euclidean geometry. But it has no privileged status just because of its close relationship to Euclid V. It may turn out to be true, or it may not.

Predicates are partial sentences which become complete when a subject is added. They are often represented as functions, whose domain is some agreed-upon universe of objects and whose range is the set {True, False}.

Predicate	Symbol	Truth Value	Proposition	Symbolic Form	Truth Value
is mortal	M()	Undefined	Socrates is mortal	M(Socrates)	True
likes chocolate	L()	Undefined	Brad likes chocolate	L(Brad)	Depends on Brad
is a perfect square	SQ()	Undefined	5 is a perfect square	SQ(5)	False

Quantifiers are the tricky part. There are two quantifiers, the *universal* quantifier “For all” (\forall) and the *existential* quantifier “there exists” (\exists). (In case those symbols didn’t come out right, for all is an upside-down A, and there exists is a backwards E.) Let P be any predicate. Then

- $\forall x P(x)$ is true if the predicate P is true for all x in some agreed-upon universe
- $\exists x P(x)$ is true if the predicate P is true for at least one x in the agreed-upon universe

If we have agreed before hand to limit our universe to, say, real numbers, then we can make statements like:

- $\forall x(x^2 \geq 0)$ For all x, x squared is greater than or equal to zero. (True)
- $\exists x(x^2 = -4)$ There is a (real) number whose square is -4. (False)
- $\exists x(x^2 = 3)$ There is a (real) number whose square is 3 (True)

Notice that the truth value of the first two statements would be different if we had agreed that x could represent any complex number. The truth value of the third statement would change if we restricted x to rational numbers.

The proposition “All men are mortal” can be rendered thus:

- $\forall x(\text{Man}(x) \Rightarrow \text{Mortal}(x))$

which says “For all things x, if x is a man, then x is mortal.” If we have previously agreed that x will only represent men, then we could shorten this to:

- $\forall x(\text{Mortal}(x))$

Most mathematical theorems are universally quantified statements. For instance, “The sum of the interior angles of a triangle is 180 degrees.” is a statement about *all* triangles. Its structure is really

- For all x, if x is a triangle, then the sum of the interior angles of x is 180 degrees.

Or, if we have a prior agreement that our conversation is limited to triangles, we can say

- For all x, the sum of the interior angles of x is 180 degrees.

When you negate a quantified statement, the quantifier changes. For instance, consider what it would take to show that this statement is false:

- All crows are black
- $\forall x(\text{Black}(x))$ (If we’ve agreed that x can only represent crows)
- $\forall x(\text{Crow}(x) \Rightarrow \text{Black}(x))$ (If x has not been restricted to crows)

If someone claims that all crows are black, you could prove her wrong by simply producing *one* non-black crow. That’s all it takes, just the existence of one crow which is not black. Thus the negation of “All crows are black”—the statement which is true exactly if the original statement is false—is

- At least one crow is not black (or, some crows are not black)
- $\exists x(\sim \text{Black}(x))$ (If x can only represent crows)
- $\exists x(\text{Crow}(x) \ \& \ \sim \text{Black}(x))$ (Unrestricted x)

The third form follows the same pattern as the second form—the quantifier changes from universal to existential, and the statement inside the quantifier gets negated. Recall that earlier we found that the negation of $(P \Rightarrow Q)$ was $(P \ \& \ \sim Q)$. That is exactly what’s happening inside the parentheses in the third form.

The same change of quantifier happens when you negate an existentially quantified statement.

Symbolic Form	English Form	Negation(Symbolic)	Negation (English)
$\forall x(\text{Man}(x) \Rightarrow \text{Mortal}(x))$	All men are mortal.	$\exists x(\text{Man}(x) \ \& \ \sim \text{Mortal}(x))$	There is an immortal man.
$\exists x(P(x) \ \& \ H(x))$	Some politicians are honest.	$\forall x(P(x) \Rightarrow \sim H(x))$ $\forall x(\sim P(x) \ \text{or} \ \sim H(x))$ $\forall x(H(x) \Rightarrow \sim P(x))$	All politicians are dishonest. (Three equivalent forms given.)

Exercise: Write each of the following statements in symbolic form two ways – first with an assumed restriction on the domain of the variable (specify what this restriction is), and then with no such restriction. Then write the symbolic negation of each form, translate the negation into English, and decide whether you really have the negation of the original English sentence.

- All doctors are men
- Some people are mean
- All politicians are scoundrels
- French people are snobs
- No shirt, no shoes, no service
- In a triangle, the greatest angle is opposite the greatest side
- Differentiable functions are continuous

Euclid’s Fifth Postulate – A Closer Look

Looking back at Euclid V, we can see that it really has quantifiers implicitly included.

- “If two lines are crossed by a transversal, and the interior angles on one side of the transversal sum to less than 180 degrees, then the lines meet on that side of the transversal.”

We understand from context that this is supposed to be a universal claim, which applies to *all cases* in which two lines are crossed by a transversal. Here is one way to explicitly represent this:

$$\forall l, m, t \left[\left(\begin{array}{l} l, m, t \\ \text{are lines} \end{array} \right) \& \left(\begin{array}{l} t \text{ crosses} \\ l \text{ and } m \end{array} \right) \& \left(\begin{array}{l} \text{the interior angles on} \\ \text{one side of } t \text{ sum} \\ \text{to less than } 180^\circ \end{array} \right) \Rightarrow \left(\begin{array}{l} l \text{ and } m \text{ meet on} \\ \text{that side of } t \end{array} \right) \right]$$

We can see that the hypothesis is a conjunction of three terms, of the form $(A \& B \& C)$. The negation of this conjunction is $\sim (A \& B \& C)$, which is equivalent to $(\sim A \text{ or } \sim B \text{ or } \sim C)$. This hypothesis could fail to be true in different ways, depending on which of its component statements fail. If we represent the structure of Euclid V as $\forall l, m, t((A \& B \& C) \Rightarrow D)$, then the contrapositive is

$$\forall l, m, t(\sim D \Rightarrow (\sim A \text{ or } \sim B \text{ or } \sim C)).$$

Finding the Logical Structure in a Statement

In any axiomatic system, the axioms are understood to be part of the hypothesis, but are not explicitly stated. If we are working in Euclidean geometry, for instance, then the statement “If two parallel lines are crossed by a transversal, then alternate interior angles are congruent,” really means “If two parallel lines are crossed by a transversal **and** all the axioms of Euclidean geometry are true, then alternate interior angles are congruent.”

Often a theorem contains no explicitly stated hypothesis at all. For instance, in Euclidean geometry there is a theorem which says “The sum of the interior angles of any triangle is 180 degrees.” (Or “is equal to the sum of two right angles,” if you want to state this in a form Euclid would recognize.) From context, we know that this is a theorem of Euclidean geometry, so its logical structure is really “If all the axioms of Euclidean geometry hold, then the sum of the interior angles of any triangle is 180 degrees.”

Proof Strategies

- **Proof by contradiction**

A *contradiction* is a statement which is false in all circumstances. The classic form of a contradiction is $P \& \sim P$, where P can be any proposition.

A *tautology* is a statement which is true in all circumstances. The classic form of a tautology is $P \text{ or } \sim P$, where P can be any proposition. You can easily verify

with a 2-line truth table that $P \& \sim P$ is always false, and that P or $\sim P$ is always true.

The technique of proof by contradiction starts with the apparently ridiculous step of *assuming the negation of what you want to prove!* That is, if we wish to prove A , we start by supposing $\sim A$. Throughout the proof, you must carefully balance in your mind this provisional assumption, saying at each step “Although I do not really believe the statement $\sim A$, I am, for the moment, accepting that it is true.” It is rather like a conversation in which one might say “While I don’t share your belief that we are all descended from Martians, I’d like to hear what conclusions follow from this belief.”

The structure of a proof by contradiction relies on the fact that the truth value of any implication $B \Rightarrow A$ is the same as the truth value of its contrapositive $\sim A \Rightarrow \sim B$. A *direct proof* would start from some statement B which we know to be true, and lead us step by step to the conclusion A . A proof by contradiction (or *indirect proof*) starts instead by provisionally assuming $\sim A$, and showing that if we believe $\sim A$, we must also believe $\sim B$. At this point, we reconnect with reality, and remember that we know B is true.

In practice, B will be either the axioms of our system (and thus the body of previously proved theorems as well), or the axioms of our system plus some additional hypothesis which is part of the theorem. In other words, B will be definitely true—a tautology. Hence $\sim B$ will be definitely false—a contradiction.

Example: The most famous proof by contradiction is the proof that $\sqrt{2}$ is irrational. This proof depends upon the following facts about the rational numbers:

- Every rational number can be written “in lowest terms,” where the numerator and denominator have no common factor (other than 1, which we never count as a factor).
- The square of an even number is even, and the square of an odd number is odd.

The proof goes thus:

Suppose that $\sqrt{2} = \frac{m}{n}$, where m and n are natural numbers. (Here we are assuming the negation of what we wish to prove.)

Then we can divide through both m and n by any factors they might have in common, reducing the fraction to lowest terms p/q , where p and q have no factors in common. Then we have $\sqrt{2} = p/q$. Multiplying both sides by q , we obtain $\sqrt{2}q = p$. Squaring both sides, we get $2q^2 = p^2$. Now the left-hand side of this equation is even, since it is twice an integer, so the right-hand side must also be even. The only way for p^2 to be even is if p is even, so we must be able to write

$p = 2k$, for some integer k . Replacing p by $2k$, we get $2q^2 = (2k)^2 = 4k^2$. Dividing both sides by 2 gives $q^2 = 2k^2$. Since the right side is even, the left side must be even, so q must be even. But now we have a contradiction! p and q are both even, yet p/q was supposed to be in lowest terms. Seeing that our original assumption, that $\sqrt{2}$ was rational, has led to a contradiction, we reject that original assumption, and conclude that $\sqrt{2}$ is irrational.

- **When the hypothesis is a conjunction**

To prove a statement of the form $(A \& B) \Rightarrow C$, we assume that A and B are both true, and try to find a line of reasoning which leads to C . Note that we need not prove either A or B , but can freely assume them.

- **When the hypothesis is a disjunction**

To prove a statement of the form $(A \text{ or } B) \Rightarrow C$, we generally will need two separate proofs. The reason for this is that we cannot be sure that A is true, nor can we be certain that B is true. We know, however, that one or the other of them is true. So if we can show that either A or B is sufficient to prove C , we will be done.

Exercise: Prove that $(A \text{ or } B) \Rightarrow C$ is equivalent to $(A \Rightarrow C) \& (B \Rightarrow C)$. We prove statements of the first form by translating them to the second form, and proving both parts of the conjunction.

- **Proof by cases**

Proof by cases is a simple extension of the preceding situation. It starts by dividing the situation into a number of cases, and then proving the desired result in each of the cases separately.

Examples:

- We wish to prove Q . Given any well-formed proposition P , it must be true that P is either true or false. So we say “Either P or $\sim P$. If P , then.... (argument leading to Q). If $\sim P$, then ... (different argument leading to Q).”
- We wish to prove some statement which involves a real number x .
 - Case 1: $x > 0$. (Some proof which works for positive numbers.)
 - Case 2: $x = 0$. (Some proof which works when $x = 0$.)
 - Case 3: $x < 0$. (Some proof which works for negative numbers.)

Caution: It is easy to cheat without meaning to, when you divide the world into cases. Typical bad starts include:

- Case 1: $x < y$, Case 2: $x > y$. (This omits all cases in which x and y are equal.)
- Case 1: x and y are positive, Case 2: x and y are negative. (This omits all cases in which one number is positive and the other negative.)

“There are two kinds of people in the world: those who divide people into categories, and those who don’t.”

- **When the conclusion is a conjunction**

The situation here is clear enough. You need to prove both parts of the conjunction.

- **When the conclusion is a disjunction**

To prove a disjunction, it suffices to prove either one of the component parts. If we are trying to prove $(A \text{ or } B)$, we might proceed by cases. In one case we find ourselves able to prove A , in the other case B . In fact, the structure of the conclusion we are aiming for often suggests a way to break the proof into cases.

Example: Every prime number is either equal to 2, or odd.

$\forall x(x \text{ is prime} \Rightarrow (x = 2) \text{ or } (x \text{ is odd}))$. Proceed by cases: If $x = 2$ we are done immediately, so most of the work goes into the case $x \neq 2$.

Recall that an implication is equivalent to a disjunction. $A \Rightarrow B$ has the same truth value as $\sim A \text{ or } B$. If we are trying to prove the disjunction $A \text{ or } B$, we can recast it as the implication $\sim A \Rightarrow B$. (Or, if we want, $\sim B \Rightarrow A$.) Then we follow the process for proving an implication.

- **When the conclusion is an implication**

Assume the hypothesis portion and prove the conclusion. This depends on the fact that $A \Rightarrow (B \Rightarrow C)$ is equivalent to $(A \& B) \Rightarrow C$. (Exercise: prove this, both by constructing an 8-line truth table, and by converting implications to disjunctions, etc.)

It may be easier to prove the contrapositive of the desired conclusion. For instance, the definition of a 1-to-1 function is:

Definition: $f : A \rightarrow B$ is called 1-to-1 (or *injective*) if $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$.

If we want to prove that a given function f is 1-to-1, we might start by assuming $f(a_1) = f(a_2)$, and try to prove that $a_1 = a_2$. In this case, we would be proving the contrapositive of the condition required for f to be 1-to-1. (NOTE: Since a statement and its contrapositive are equivalent, we could equally well have used the contrapositive as the definition of a 1-to-1 function.)

- **Proving Existence**

If it is possible to produce an example of the thing whose existence is in question, this can be a very easy proof. We can prove the statement $\exists x((x \text{ is an integer}) \& (x \text{ divides } 24523))$ by producing the number 137.

Sometimes it is not possible to produce (or “demonstrate”) the thing whose existence is in question, and in this case a proof of existence can be very difficult.

One example is the Intermediate Value Theorem for functions from the real numbers to the real numbers: if a continuous function f has both negative and positive values on an interval $[a, b]$, then there exists a number c in the interval such that $f(c) = 0$. This is true, but the proof requires some non-trivial topology.

- **Proving Uniqueness**

To prove that there is a unique object x such that some proposition $P(x)$ is true, suppose that $P(x)$ and $P(y)$ are both true, and prove that $x = y$. It is often easiest to develop this as a proof by contradiction, adding to your initial assumptions the hypothesis that $x \neq y$ and deriving a contradiction. Afterward, though, you should see if you can restructure the proof as a direct proof that $x = y$, as direct proofs of uniqueness are often easier to grasp.

- **Proving a Universally Quantified Statement**

Proving a statement of the form $\forall x(P(x))$, or $\forall x(P(x) \Rightarrow Q(x))$, can be a bit tricky. We need to know enough about x to prove something about it, and yet leave it sufficiently unspecified that our proof will apply to all members of the agreed-upon universe. The best approach is to assume nothing about x other than what we are given. For instance, in proving $\forall x(P(x) \Rightarrow Q(x))$, we would take $P(x)$ as given, and try to prove $Q(x)$ without introducing any additional assumptions. If we are successful, we know that our proof is perfectly general—it applies equally well to all x .

Exercise: Let x and y be restricted to the set of real numbers. Consider the partially defined predicate $P(x,y) = (x < y \Rightarrow x^2 < y^2)$. It is impossible to assign a truth value to $P(x,y)$ without imposing some kind of quantifiers on x and y . In fact, there are eight distinct ways to do this. One example is $\forall x \exists y (x < y \Rightarrow x^2 < y^2)$, or $\forall x \exists y P(x,y)$. The other seven are obtained by rearranging the quantifiers in front of P . Write out all eight statements. Decide whether each one is true or false (where x and y are understood to be real numbers). If a statement is true, prove it. If false, write down and prove its negation.